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Nonlinear differential difference equations as Bäcklund transformations

D Levi†

The Rockefeller University, New York, NY 10021 USA

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Abstract. In this paper, one shows that the best known nonlinear differential difference equations associated with the discrete Schrödinger spectral problem and also with the discrete Zakharov–Shabat spectral problem can be interpreted as Bäcklund transformations for some continuous nonlinear evolution equations.

1. Introduction

In a recent article Levi and Benguria (1980) have shown that there exists a well defined relation between Bäcklund transformations (BTs) associated with a class of nonlinear evolution equations (NEEs) and some nonlinear differential difference equations (NDDEs). Therein it was proved that, if

$$\psi_x(x, \lambda) = U(q(x), \lambda)\psi(x, \lambda)\ddagger \tag{1}$$

is the linear Schrödinger spectral problem (SP) corresponding to the 'potential' $q(x)$, then there exists a matrix W, depending on the 'potential' $q(x)$ and on a different 'potential' $q'(x)$

$$W = W(q(x), q'(x), \lambda)$$

such that the usual BTs read

$$W_x(q(x), q'(x), \lambda) = U(q'(x), \lambda)W(q(x), q'(x), \lambda) - W(q(x), q'(x), \lambda)U(q(x), \lambda). \tag{2}$$

Equation (2) is obtained as the compatibility condition between the SP (1) and the following equation

$$\psi'(x, \lambda) = W(q(x), q'(x), \lambda)\psi(x, \lambda) \tag{3}$$

where $\psi'(x, \lambda)$ is the wavefunction of the SP (1) corresponding to the 'potential' $q'(x)$. By defining $q(x) = q(x, n)$ and $q'(x) = q(x, n+1)$, where n varies on the integers, equation (2) can be thought of as an NDDE. Levi and Benguria (1980) proved this property for the BTs associated with the matrix Schrödinger SP, as this SP seemed to the authors to be the most general available to them. This property can also be shown, in a quite similar way, to hold for the Zakharov–Shabat SP§.

† Present Address: Istituto di Fisica, Università di Roma, Piazzale Aldo Moro 2, 00185 Roma, Italy.
‡ Here and in the following, f_x represents $\partial f/\partial x$, and $\dot{f}$ represents $\partial f/\partial t$.
§ The discrete sine-Gordon equation considered by Orfanidis (1978) and Levi *et al* (1980) is just an example of where a BT of the Zakharov–Shabat SP is interpreted as an NDDE.

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Thus the existence of a matrix $W(q, q', \lambda)$, which is the generator of the BTs for a given SP, appears as a quite general property.

In this paper one wants also to show how the best known NDDEs can be cast in this scheme. In the literature on NDDEs one can find two discrete SPs which have been thoroughly discussed: the discrete Schrödinger SP which gives rise to the Toda lattice hierarchy (Toda 1975, Flaschka 1974 and Dodd 1978) and its matrix generalisation (Bruschi *et al* 1980, 1981), and the discrete Zakharov–Shabat SP (Ablowitz and Ladik 1975, Chiu and Ladik 1977).

The discrete Schrödinger SP reads†:

$$v(n-1, t, \lambda)+b(n, t)v(n, t, \lambda)+a(n, t)v(n+1, t, \lambda)=\lambda v(n, t, \lambda). \tag{4}$$

The simplest associated NDDE that one obtains from the generalised Wronskian technique is (Bruschi *et al* 1981)

$$\begin{aligned} \dot a(n, t) &= a(n, t)[b(n+1, t)-b(n, t)] \\ \dot b(n, t) &= a(n, t)-a(n-1, t) \end{aligned} \tag{5}$$

when the reflection coefficient evolves according to

$$\dot R(z, t)=-\mu R(z, t)$$

with z defined by

$$\lambda = z+z^{-1} \tag{6}$$

and μ defined by $\mu = z - z^{-1}$.

By the definitions

$$\begin{aligned} a(n, t) &= \exp\{-[Q(n+1, t)-Q(n, t)]\} \\ b(n, t) &= -\dot Q(n, t) \end{aligned}$$

equation (5) can be written as the Toda lattice equation

$$\ddot Q(n, t)=\exp[-(Q(n, t)-Q(n-1, t))]-\exp[-(Q(n+1, t)-Q(n, t))].$$

From equation (4), by going to a higher order in the hierarchy of NDDEs and setting $b(n, t)=0$, we can obtain the equation

$$\dot a(n, t)=a(n, t)[a(n+1, t)-a(n-1, t)] \tag{7}$$

which corresponds to the following evolution of the reflection coefficient

$$\dot R(z, t)=-\mu\lambda R(z, t).$$

Equation (7) is called the generalised Volterra equation (Wadati 1976) and describes a system of an infinite number of interacting species.

The discrete Zakharov–Shabat SP, in the most general formulation proposed by Ablowitz and Ladik (1975), reads

$$\begin{aligned} v_1(n+1, z) &= zv_1(n, z)+Q(n)v_2(n, z)+S(n)v_2(n+1, z) \\ v_2(n+1, z) &= z^{-1}v_2(n, z)+R(n)v_1(n, z)+T(n)v_1(n+1, z) \end{aligned} \tag{8}$$

and contains four independent 'potentials', $Q(n, t)$, $R(n, t)$, $S(n, t)$ and $T(n, t)$. The

† For convenience, one writes the discrete Schrödinger SP in the form written down by Bruschi *et al* (1981), even if, for the sake of simplicity, one will consider only the abelian case.

simplest associated NDDE is

$$\begin{aligned}\dot{Q}(n,t)&=[1-Q(n,t)R(n,t)][S(n,t)-S(n-1,t)]\\ \dot{R}(n,t)&=[1-Q(n,t)R(n,t)][T(n,t)-T(n-1,t)]\\ \dot{S}(n,t)&=[1-S(n,t)T(n,t)][Q(n+1,t)-Q(n,t)]\\ \dot{T}(n,t)&=[1-S(n,t)T(n,t)][R(n+1,t)-R(n,t)]\end{aligned} \tag{9}$$

which corresponds to the following evolution of the reflection coefficient

$$\dot{R}(z,t)=-\mu R(z,t)$$

and contains as subcases the Toda lattice (for $R(n,t)=0$, $T(n,t)=1$, $Q(n,t)=-b(n,t)$ and $S(n,t)=1-a(n,t)$) and the self-dual network (for $R(n,t)=\pm Q(n,t)=I(n,t)$ and $S(n,t)=\pm T(n,t)=V(n,t)$) (Ablowitz and Ladik 1975). From equation (8), by going to a higher order in the hierarchy of NDDEs and setting $S(n,t)=T(n,t)=0$, one can obtain the equation

$$\begin{aligned}\dot{Q}(n,t)&=[1-Q(n,t)R(n,t)][Q(n+1,t)-Q(n-1,t)]\\ \dot{R}(n,t)&=[1-Q(n,t)R(n,t)][R(n+1,t)-R(n-1,t)]\end{aligned} \tag{10}$$

which corresponds to the following evolution of the reflection coefficient

$$\dot{R}(z,t)=-\mu\lambda R(z,t)$$

and contains as subcases the discrete analogue of the nonlinear Schrödinger equation (for $R(n,t)=\pm Q^*(n,t)$, where by Q^* one means the complex conjugate of Q) and of the modified Korteweg–de Vries equation (for $R(n,t)=\pm Q(n,t)$).

In § 2 one will show in all details how one can construct, for the Toda lattice equation (5), the SP whose associated NEEs have the Toda itself as a BT. Section 3 is devoted to a brief account of the results one obtains for the Volterra equation (7) while § 4 contains the results related to equations (9) and (10).

2. The Toda lattice as a BT

The Toda lattice equation (5) can be obtained as the compatibility condition between the discrete SP (4) and the equation

$$\dot{v}(n,t,\lambda)=a(n,t)v(n+1,t,\lambda) \tag{11}$$

which gives the evolution of the wavefunction $v(n,t,\lambda)$. Defining the vector $\boldsymbol{v}(n,t,\lambda)$ of components $(v(n,t,\lambda), v(n+1,t,\lambda))$ equations (4) and (11) can be written as matrices of rank 2, equation (4) being a functional equation of second order in the discrete variable n. Introducing the variable x instead of the time t, for notational convenience, equations (4) and (11) read

$$\boldsymbol{v}_x(n,x,\lambda)=\begin{pmatrix}0 & a(n,x)\\ -1 & \lambda-b(n+1,x)\end{pmatrix}\boldsymbol{v}(n,x,\lambda) \tag{12}$$

$$\boldsymbol{v}(n-1,x,\lambda)=\begin{pmatrix}\lambda-b(n,x) & -a(n,x)\\ 1 & 0\end{pmatrix}\boldsymbol{v}(n,x,\lambda). \tag{13}$$

Now one introduces the continuous SP

$$v_x(x,\lambda)=\begin{pmatrix}0 & 1+q(x)\\ -1 & \lambda+r(x)\end{pmatrix}v(x,\lambda)=U(q(x),r(x),\lambda)v(x,\lambda). \tag{14}$$

By introducing into the SP (14) a parametrical dependence on a discrete variable n in such a way that one can identify

$$\begin{aligned} q(x)&=a(n,x)-1\\ r(x)&=-b(n+1,x)\end{aligned} \tag{15}$$

one recovers equation (12)†.
By defining $v'(x,\lambda)=v(n-1,x,\lambda)$ and

$$q'(x)=a(n-1,x)-1 \qquad r'(x)=-b(n,x) \tag{16}$$

equation (13) reads

$$v'(x,\lambda)=\begin{pmatrix}\lambda+r'(x) & -1-q(x)\\ 1 & 0\end{pmatrix}v(x,\lambda) \tag{17}$$

and the Toda lattice equation (5), being the compatibility condition between equation (14) and equation (17), which are of the form of equation (1) and equation (3), is thus the BT for the NEEs associated with equation (14) interpreted as a SP. As a BT the Toda equation reads

$$\begin{aligned} q_x(x)&=[1+q(x)][r'(x)-r(x)]\\ r'_x(x)&=q'(x)-q(x).\end{aligned} \tag{18}$$

In the following part of this section one will obtain the NEEs associated with the SP (14) together with its BTs. For $q(x)$ and $r(x)$ vanishing asymptotically, the matrix

$$U^0(\lambda)=\lim_{|x|\to\infty} U(q(x),r(x),\lambda)$$

is constant but not diagonal; thus, introducing the constant, eigenvalue dependent, matrix K

$$K=(\sigma_0+z\sigma_1)/(1+z^2)^{1/2} \tag{19}$$

where z is defined by equation (6) with respect to λ, by σ_0 one means the identity matrix of rank 2 and σ_1 is the first Pauli matrix

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

the matrix $U^0(\lambda)$ can be diagonalised, thus enabling one to define in a unique and simple way the associated Jost functions. Under the transformation K the SP (14) reads

$$\begin{aligned}\psi_x(x,z)&=\begin{pmatrix} z+z(q-zr)/(1-z^2) & (q-zr)/(1-z^2)\\ z(r-zq)/(1-z^2) & z^{-1}+(r-zq)/(1-z^2)\end{pmatrix}\psi(x,z)\\ &=\bar{U}(q(x),r(x),z)\psi(x,z)\end{aligned} \tag{20}$$

† Recalling, from Bruschi *et al* (1980), that as $|n|\to\infty$ $a(n)\to 1$ and $b(n)\to 0$, the functions $q(x)$ and $r(x)$ have been defined in equation (15) in such a way that both vanish asymptotically.

with $\psi(x, z)$ defined by

$$\psi(x, z) = K^{-1} v(x, \lambda). \tag{21}$$

$\bar{U}^0(z)$ reads:

$$U^0(z) = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix};$$

it is diagonal and depends only on the 'eigenvalue' z. To introduce a set of Jost functions for the SP (20) with the required analytical properties, one introduces the variable $y = \mathrm{i}x$, in such a way that the SP (20) reads

$$\psi_y(y, z) = -\mathrm{i}\bar{U}(q(y), r(y), z)\psi(y, z). \tag{22}$$

For the SP (22) one can introduce, in a similar way as for the Zakharov–Shabat SP (Ablowitz 1978), the following two sets of Jost functions

$$\psi(y, z) \underset{y\to+\infty}{\sim} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\{-\mathrm{i}yz^{-1}\} \qquad \varphi(y, z) \underset{y\to-\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\{-\mathrm{i}yz\}$$

$$\bar{\psi}(y, z) \underset{y\to+\infty}{\sim} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\{-\mathrm{i}yz\} \qquad \bar{\varphi}(y, z) \underset{y\to-\infty}{\sim} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp\{-\mathrm{i}yz^{-1}\}.$$

One can easily prove (a detailed discussion of the SP (22) and of the similar ones that will be met in the following sections together with the solution of the inverse problem will be given in a subsequent article) that $\psi(y, z)$ and $\varphi(y, z)$ (respectively $\bar{\psi}(y, z)$ and $\bar{\varphi}(y, z)$) can be continued analytically in the upper (respectively lower) half of the complex plane of the 'eigenvalue' z. One can define the Wronskian of two solutions $\psi_1(y, z)$ and $\psi_2(y, z)$ of equation (22) as

$$\mathrm{Wr}(\psi_1, \psi_2) = \mathrm{i} \exp\left[\mathrm{i}\left(\lambda y - \int_y^\infty \mathrm{d}\xi\, r(\xi)\right)\right] \psi_1^{\mathrm{T}}(y, z)\sigma_2\psi_2(y, z) \tag{23}$$

where by ψ^{T} one means the transpose of ψ and, as λ and $r(y)$ are taken to be real, equation (23) is asymptotically well defined. By direct calculation one gets

$$\mathrm{Wr}_y(\psi_1, \psi_2) = 0$$

which implies that the Wronskian defined in equation (23) is constant in y. In particular one has

$$\mathrm{Wr}(\psi(y, z), \bar{\psi}(y, z)) = -1$$

$$\mathrm{Wr}(\varphi(y, z), \bar{\varphi}(y, z)) = -\exp\left(-\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}\xi\, r(\xi)\right).$$

From the constancy of the Wronskian and the linear independence of $\psi(y, z)$ and $\bar{\psi}(y, z)$ one can write

$$\begin{aligned} \varphi(y, z) &= a(z)\bar{\psi}(y, z) + b(z)\psi(y, z) \\ \bar{\varphi}(y, z) &= -\bar{a}(z)\psi(y, z) + \bar{b}(z)\bar{\psi}(y, z) \end{aligned} \tag{24}$$

where $a(z) = \mathrm{Wr}(\varphi, \psi)$ (respectively $\bar{a}(z) = \mathrm{Wr}(\bar{\varphi}, \bar{\psi})$) can be continued in the upper (respectively lower) half of the complex z plane. By calculating $\mathrm{Wr}(\varphi, \bar{\varphi})$ one gets the following relation between the coefficients of equation (24)

$$a(z)\bar{a}(z)+\bar{b}(z)b(z)=\exp\left(-\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{d}\xi\, r(\xi)\right)$$

which, for $r(y)=0$, reduces to the usual unitarity relation of the Schrödinger SP.

Defining the reflection and transmission coefficients as

$$\begin{aligned} R(z)&=b(z)/a(z) & T(z)&=1/a(z)\\ \bar{R}(z)&=\bar{b}(z)/\bar{a}(z) & \bar{T}(z)&=1/\bar{a}(z) \end{aligned} \tag{25}$$

one can obtain the following asymptotic behaviour for the matrix solution $\Psi(y,z)$ of equation (22), constructed by putting together two independent solutions of the SP (22)

$$\begin{aligned} \Psi(y,z) &\underset{y\to-\infty}{\sim} \begin{pmatrix} T(z)\,\mathrm{e}^{-\mathrm{i}yz} & 0\\ 0 & -\bar{T}(z)\,\mathrm{e}^{-\mathrm{i}yz^{-1}} \end{pmatrix}\\ &\underset{y\to+\infty}{\sim} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}yz} & \bar{R}(z)\,\mathrm{e}^{-\mathrm{i}yz}\\ R(z)\,\mathrm{e}^{-\mathrm{i}yz^{-1}} & -\mathrm{e}^{-\mathrm{i}yz^{-1}} \end{pmatrix}. \end{aligned} \tag{26}$$

To obtain the NEEs and the BTs one applies the technique of the generalised Wronskian (Calogero 1976), by introducing the following Wronskian

$$\mathrm{GW}(V'(x,\lambda),V(x,\lambda))=V'^{-1}(x,\lambda)F(x,\lambda)V(x,\lambda) \tag{27}$$

where by $V(x,\lambda)$ one means a nonsingular matrix solution of the SP (14) and $F(x,\lambda)$ is a generic matrix function of elements $a(x,\lambda)$, $b(x,\lambda)$, $c(x,\lambda)$ and $d(x,\lambda)$

$$F(x,\lambda)=\begin{pmatrix} a(x,\lambda) & b(x,\lambda)\\ c(x,\lambda) & d(x,\lambda)\end{pmatrix}.$$

From equation (27) one easily obtains†

$$\mathrm{GW}\Big|_{-\infty}^{+\infty}=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\begin{pmatrix} a_x-b-c(1+q') & b_x+a(1+q)-d(1+q')+b(\lambda+r)\\ c_x+a-d-c(\lambda+r') & d_x+c(1+q)+b+d(r-r')\end{pmatrix}V. \tag{28}$$

As, by equation (21), one has

$$\mathrm{GW}=\Psi'^{-1}K^{-1}FK\Psi=\Psi'^{-1}\bar{F}\Psi$$

with

$$\bar{F}=\frac{1}{1-z^2}\begin{pmatrix} a-z^2d-z(c-b) & z(a-d)+b-z^2c\\ c-z^2b-z(a-d) & d-z^2a+z(c-b)\end{pmatrix}=\begin{pmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix}$$

the LHS of equation (28) becomes, taking into account equation (26):

$$\begin{aligned} \mathrm{GW}\Big|_{-\infty}^{+\infty}=&\frac{1}{1+R'\bar{R}'}\begin{pmatrix} \bar{a}+\bar{d}\bar{R}'R & \bar{a}\bar{R}-\bar{d}\bar{R}'\\ \bar{a}R'-\bar{d}R & \bar{a}\bar{R}R'+\bar{d}\end{pmatrix}-\begin{pmatrix}\bar{a}TT'^{-1} & 0\\ 0 & \bar{d}\bar{T}\bar{T}'^{-1}\end{pmatrix}\\ &+\frac{1}{1+R'\bar{R}'}\begin{pmatrix} \bar{c}\bar{R}'\,\mathrm{e}^{-\mathrm{i}\mu y}+\bar{b}R\,\mathrm{e}^{\mathrm{i}\mu y} & \bar{c}\bar{R}\bar{R}'\,\mathrm{e}^{-\mathrm{i}\mu y}-\bar{b}\,\mathrm{e}^{\mathrm{i}y\mu}\\ \bar{b}RR'\,\mathrm{e}^{\mathrm{i}\mu y}-\bar{c}\,\mathrm{e}^{-\mathrm{i}\mu y} & -(\bar{c}\bar{R}\,\mathrm{e}^{-\mathrm{i}\mu y}+\bar{b}R'\,\mathrm{e}^{\mathrm{i}\mu y})\end{pmatrix}\\ &+\begin{pmatrix} 0 & \bar{b}\bar{T}T'^{-1}\,\mathrm{e}^{\mathrm{i}\mu y}\\ \bar{c}T\bar{T}'^{-1}\,\mathrm{e}^{-\mathrm{i}\mu y} & 0\end{pmatrix}. \end{aligned} \tag{29}$$

† Here and in the following one will omit the x and λ dependence of the matrix solution $V(x,\lambda)$ and of F and its elements a, b, c, d.

To give some meaning to equation (29) one has two possible choices for $F(x, \lambda)$.

(1) All elements of $F(x, \lambda)$ vanish as $|x| \to \infty$ so that also $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ vanish and thus

$$\mathrm{GW}\,|_{-\infty}^{+\infty} = 0. \tag{30}$$

(2) a, b, c and d do not depend on x, but are such that $\bar{b} = \bar{c} = 0$; in this case one must have

$$a(\lambda) = d + c\lambda \qquad b = -c. \tag{31}$$

In case (1) from equations (28) and (30), after some manipulation, one gets the following relations

$$\lambda \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1} \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} V = \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}$$
$$\times \begin{pmatrix} 0; & -b_x - br + (1+q)\int_x^\infty \mathrm{d}\xi[b - c(1+q')] + (1+q')\,\mathrm{e}^{Y(x)} \int_x^\infty \mathrm{d}\xi[b - c(1+q)]\,\mathrm{e}^{-Y(\xi)} \\ c_x - cr' + \int_x^\infty \mathrm{d}\xi[b - c(1+q')] + \mathrm{e}^{Y(x)} \int_x^\infty \mathrm{d}\xi[b - c(1+q)]\,\mathrm{e}^{-Y(\xi)}; & 0 \end{pmatrix} V \tag{32}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1} \begin{pmatrix} a_x & a(1+q) - d(1+q') \\ a - d & d_x + d(r - r') \end{pmatrix} V = 0 \tag{33}$$

with

$$Y(x) = \int_x^\infty \mathrm{d}\xi[r(\xi) - r'(\xi)]\,.$$

Equation (32) defines an operator $\mathscr{L}$ such that

$$\eta\lambda \begin{pmatrix} b \\ c \end{pmatrix} = \eta\mathscr{L} \begin{pmatrix} b \\ c \end{pmatrix}$$

$$\mathscr{L} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -b_x - br + (1+q)\int_x^\infty \mathrm{d}\xi[b - c(1+q')] + (1+q')\,\mathrm{e}^{Y(x)} \int_x^\infty \mathrm{d}\xi\, \mathrm{e}^{-Y(\xi)}[b - c(1+q)] \\ c_x - cr' + \int_x^\infty \mathrm{d}\xi[b - c(1+q')] + \mathrm{e}^{Y(x)} \int_x^\infty \mathrm{d}\xi\, \mathrm{e}^{-Y(\xi)}[b - c(1+q)] \end{pmatrix} \tag{34}$$

with η such that

$$\eta \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

while equation (33) provides the necessary rules for writing a given matrix in a pre-required form, i.e. to make it off-diagonal or with the first column zero. For example, from equation (33) one obtains

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1} \eta \begin{pmatrix} b \\ c \end{pmatrix} V = \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1} \begin{pmatrix} 0 & b - c(1+q') \\ 0 & c_x + c(r - r') \end{pmatrix} V$$

i.e. one can define the operator $\mathscr{M}$ such that

$$\mathscr{M} \begin{pmatrix} b \\ c \end{pmatrix} = \rho \begin{pmatrix} b - c(1+q') \\ c_x + c(r - r') \end{pmatrix} \tag{35}$$

with ρ such that

$$\rho\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}0 & \alpha\\ 0 & \beta\end{pmatrix}.$$

In case (2), from equations (28) and (31), one gets:

$$\frac{1}{1+R'\bar{R}'}\begin{pmatrix}1+\bar{R}'R & -(\bar{R}'-\bar{R})\\ R'-R & 1+\bar{R}R'\end{pmatrix}-\begin{pmatrix}TT'^{-1} & 0\\ 0 & \bar{T}\bar{T}'^{-1}\end{pmatrix}=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\rho\begin{pmatrix}-(q'-q)\\ -(r'-r)\end{pmatrix}V \quad (36)$$

$$\frac{1}{1+R'\bar{R}'}\begin{pmatrix}z^{-1}+z\bar{R}'R & z^{-1}\bar{R}-z\bar{R}'\\ z^{-1}R'-zR & z+z^{-1}\bar{R}R'\end{pmatrix}-\begin{pmatrix}z^{-1}TT'^{-1} & 0\\ 0 & z\bar{T}\bar{T}'^{-1}\end{pmatrix}$$
$$=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\eta\begin{pmatrix}-q_x-(1+q)[r-\int_x^\infty\mathrm{d}\xi(q-q')]\\ -r'+\int_x^\infty\mathrm{d}\xi(q-q')\end{pmatrix}V \quad (37)$$

where, to get equation (36), one has set $a=d=1$, $b=c=0$ and to get equation (37) one has set $a=\lambda$, $b=-1$, $c=1$ and $d=0$ and has used equations (33) and (34).

From equations (34)–(37) one can construct the hierarchy of NEEs and BTs that can be associated with the SP (14).

Setting

$$\begin{aligned}
&q(x)=q(x,t) && q'(x)=q(x,t)+\dot{q}(x,t)\,\mathrm{d}t\\
&r(x)=r(x,t) && r'(x)=r(x,t)+\dot{r}(x,t)\,\mathrm{d}t\\
&R(z)=R(z,t) && R'(z)=R(z,t)+\dot{R}(z,t)\,\mathrm{d}t\\
&\bar{R}(z)=\bar{R}(z,t) && \bar{R}'(z)=\bar{R}(z,t)+\dot{\bar{R}}(z,t)\,\mathrm{d}t\\
&\bar{T}(z)=\bar{T}(z,t) && \bar{T}'(z)=\bar{T}(z,t)+\dot{\bar{T}}(z,t)\,\mathrm{d}t\\
&T(z)=T(z,t) && T'(z)=T(z,t)+\dot{T}(z,t)\,\mathrm{d}t
\end{aligned} \quad (38)$$

equations (34)–(37) become:

$$\lambda\begin{pmatrix}b\\ c\end{pmatrix}=\boldsymbol{L}\begin{pmatrix}b\\ c\end{pmatrix}$$
$$\boldsymbol{L}\begin{pmatrix}b\\ c\end{pmatrix}=\begin{pmatrix}-b_x-br+2(1+q)\int_x^\infty\mathrm{d}\xi[b-c(1+q)]\\ c_x-cr+2\int_x^\infty\mathrm{d}\xi[b-c(1+q)]\end{pmatrix} \quad (39)$$

$$\boldsymbol{M}\begin{pmatrix}b\\ c\end{pmatrix}=\rho\begin{pmatrix}b-c(1+q)\\ c_x\end{pmatrix} \quad (40)$$

$$\begin{pmatrix}\dot{T}T^{-1}-\dfrac{\dot{R}\bar{R}}{1+R\bar{R}} & \dfrac{-\dot{\bar{R}}}{1+R\bar{R}}\\ \dfrac{\dot{R}}{1+R\bar{R}} & \dot{\bar{T}}\bar{T}^{-1}-\dfrac{\dot{\bar{R}}R}{1+R\bar{R}}\end{pmatrix}=\int_{-\infty}^{+\infty}\mathrm{d}x\,V^{-1}\rho\begin{pmatrix}-\dot{q}\\ -\dot{r}\end{pmatrix}V \quad (41)$$

$$\frac{\mu}{1+R\bar{R}}\begin{pmatrix}R\bar{R} & -\bar{R}\\ -R & -R\bar{R}\end{pmatrix}=\int_{-\infty}^{+\infty}\mathrm{d}x\,V^{-1}\eta\begin{pmatrix}-q_x-r(1+q)\\ -r\end{pmatrix}V. \quad (42)$$

From equations (39)–(42) one can assert that, if $q(x,t)$ and $r(x,t)$ evolve in time according to the following NEE

$$\rho\begin{pmatrix}\dot{q}\\ \dot{r}\end{pmatrix}+\boldsymbol{M}f(\boldsymbol{L})\begin{pmatrix}-q_x-r(1+q)\\ -r\end{pmatrix}=0 \tag{43}$$

then the corresponding reflection and transmission coefficients evolve linearly according to

$$\dot{\bar{R}}=\mu f(\lambda)\bar{R} \qquad \dot{\bar{T}}=0$$
$$\dot{R}=-\mu f(\lambda)R \qquad \dot{T}=0.$$

In the following one writes down the simpler systems of the hierarchy given by equation (43). For $f(\lambda)=1$, one has

$$\dot{q}=q_x \qquad \dot{r}=r_x,$$

for $f(\lambda)=\lambda$ one has

$$\dot{q}=-q_{xx}-2r_x-2(rq)_x \qquad \dot{r}=r_{xx}-2q_xA(r^2)_x$$

and for $f(\lambda)=\lambda^2-4$ one has

$$\begin{aligned}\dot{q}&=q_{xxx}+6qq_x+3[r^2(1+q)+rq_x]_x\\ \dot{r}&=r_{xxx}+(r^3-3rr_x+6rq)_x.\end{aligned} \tag{44}$$

It is worthwhile to notice that equation (44) contains the Korteweg–de Vries equation as a subcase, by setting $r(x, t_0)=0$.

The auto-Bäcklund transformations are obtained (Calogero 1975) by taking into account equations (34)–(37) and are given by

$$\rho p\begin{pmatrix}q'-q\\ r'-r\end{pmatrix}+\mathcal{M}f(\mathcal{L})\begin{pmatrix}-q_x-(1+q)[r-\int_x^\infty \mathrm{d}\xi(q-q')]\\ -r'+\int_x^\infty \mathrm{d}\xi(q-q')\end{pmatrix}=0 \tag{45}$$

where p is an arbitrary constant; if the 'potentials' q, r and q', r' are related by equation (45), the corresponding reflection and transmission coefficients are transformed according to the following relations:

$$\bar{R}'=\bar{R}\left(\frac{p-z^{-1}f(\lambda)}{p-zf(\lambda)}\right) \qquad \bar{T}'=\bar{T}$$
$$R'=R\left(\frac{p-zf(\lambda)}{p-z^{-1}f(\lambda)}\right) \qquad T'=T.$$

From equation (45), for $p=0$, one gets the Toda lattice equation as written in equation (18). For p arbitrary and $f(\lambda)=1$, one gets:

$$q_x=(q-q')\left[p-r'+\int_x^\infty \mathrm{d}\xi(q-q')\right]-(r-r')(1+q)$$
$$r'_x=(r-r')\left[p-r'+\int_x^\infty \mathrm{d}\xi(q-q')\right]-(q-q')$$

which, using formulae (15) and (16), can be cast in the form

$$\dot{a}(n)=[a(n)-a(n-1)]\left(p+b(n)+\int_t^\infty \mathrm{d}\xi(a(n)-a(n-1))\right)+a(n)[b(n+1)-b(n)]$$

$$\dot{b}(n) = -[b(n+1) - b(n)]\left(p + b(n) + \int_t^\infty \mathrm{d}\xi (a(n) - a(n-1))\right] - [a(n) - a(n-1)]$$

i.e. a nonlocal in time, nonlinear lattice equation.

3. The Volterra equation as a BT

Equation (7) is obtained as the compatibility condition between the SP (4) with $b(n, t) = 0$ and the following evolution equation of the wavefunction $v(n, t, \lambda)$

$$\dot{v}(n, t, \lambda) = -a(n, t)v(n, t, \lambda) + \lambda a(n, t)v(n+1, t, \lambda).$$

In matrix form, using the variable x instead of the time t, one gets:

$$\boldsymbol{v}_x(n, x, \lambda) = \begin{pmatrix} -a(n, x) & \lambda a(n, x) \\ -\lambda & \lambda^2 - a(n+1, x) \end{pmatrix} \boldsymbol{v}(n, x, \lambda) \tag{46}$$

$$\boldsymbol{v}(n-1, x, \lambda) = \begin{pmatrix} \lambda & -a(n, x) \\ 1 & 0 \end{pmatrix} \boldsymbol{v}(n, x, \lambda). \tag{47}$$

Introducing the two pairs of potentials q, r and q', r' in such a way that

$$q(x) = a(n, x) - 1 \qquad q'(x) = a(n-1, x) - 1$$
$$r(x) = a(n+1, x) - 1 \qquad r'(x) = a(n, x) - 1$$

equations (7), (46) and (47) become

$$\begin{aligned} &q(x) = r'(x) \\ &q_x(x) = (1 + q(x))(r(x) - q'(x)) \end{aligned} \tag{48}$$

$$\boldsymbol{v}_x(x, \lambda) = \begin{pmatrix} -(1+q(x)) & \lambda(1+q(x)) \\ -\lambda & \lambda^2 - r(x) - 1 \end{pmatrix} \boldsymbol{v}(x, \lambda) = U(q(x), r(x), \lambda)\boldsymbol{v}(x, \lambda) \tag{49}$$

$$\boldsymbol{v}'(x, \lambda) = \begin{pmatrix} \lambda & -(1+q(x)) \\ 1 & 0 \end{pmatrix} \boldsymbol{v}(x, \lambda) = W(q(x), r(x), q'(x), r'(x), \lambda)\boldsymbol{v}(x, \lambda) \tag{50}$$

where thus equation (48) is a BT for the hierarchy of NEEs associated with the SP (49) and $W(q, r, q', r', \lambda)$, defined by equation (50), is thus the generator of the BT (48).

Using a method similar to that in the previous section, one diagonalises the matrix U by the use of the *same* matrix $\boldsymbol{K}$ given by equation (19), after having defined the 'eigenvalue' z as in equation (6). Introducing the variable $y = \mathrm{i}x$, the SP (49) reads

$$\begin{aligned} \boldsymbol{\psi}_y(y, z) &= -\mathrm{i}\begin{pmatrix} z^2 + z^2(q-r)/(1-z^2) & (qz^{-1} - rz)/(1-z^2) \\ z^2(rz^{-1} - qz)/(1-z^2) & z^{-2} - (q-r)/(1-z^2) \end{pmatrix} \boldsymbol{\psi}(y, z) \\ &= -\mathrm{i}\bar{U}(q(y), r(y), z)\boldsymbol{\psi}(y, z). \end{aligned} \tag{51}$$

It is worthwhile to notice that the SP (51) is of the same kind as the one introduced for the Toda lattice (22) up to the proper change of z into z^2.

The Wronskian is

$$\mathrm{Wr}(\boldsymbol{\psi}_1, \boldsymbol{\psi}_2) = \mathrm{i} \exp\left\{\mathrm{i}\left[(\lambda^2-2)y - \int_y^{\infty} \mathrm{d}\xi (r-q)\right]\right\} \boldsymbol{\psi}_1^{\mathrm{T}}(y,z)\sigma_2 \boldsymbol{\psi}_2(y,z)$$

and is constant. Introducing, by analogy with the previous section, the Jost functions

$$\begin{aligned} \boldsymbol{\psi}(y,z) &\underset{y\to+\infty}{\sim} \begin{pmatrix}0\\1\end{pmatrix} \mathrm{e}^{-\mathrm{i}yz^{-2}} \qquad & \boldsymbol{\varphi}(y,z) &\underset{y\to-\infty}{\sim} \begin{pmatrix}1\\0\end{pmatrix} \mathrm{e}^{-\mathrm{i}yz^{2}} \\ \bar{\boldsymbol{\psi}}(y,z) &\underset{y\to+\infty}{\sim} \begin{pmatrix}1\\0\end{pmatrix} \mathrm{e}^{-\mathrm{i}yz^{2}} \qquad & \bar{\boldsymbol{\varphi}}(y,z) &\underset{y\to-\infty}{\sim} \begin{pmatrix}0\\-1\end{pmatrix} \mathrm{e}^{-\mathrm{i}yz^{-2}} \end{aligned} \tag{52}$$

one can define, as in equations (24) and (25), the reflection and transmission coefficients $R(z)$ and $T(z)$, and thus obtain the following asymptotics for the matrix solution of equation (51)

$$\begin{aligned} \Psi(y,z) &\underset{y\to-\infty}{\sim} \begin{pmatrix} T(z)\,\mathrm{e}^{-\mathrm{i}yz^2} & 0 \\ 0 & -\bar{T}(z)\,\mathrm{e}^{-\mathrm{i}yz^{-2}} \end{pmatrix} \\ &\underset{y\to+\infty}{\sim} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}yz^2} & \bar{R}(z)\,\mathrm{e}^{-\mathrm{i}yz^2} \\ R(z)\,\mathrm{e}^{-\mathrm{i}yz^{-2}} & -\mathrm{e}^{-\mathrm{i}yz^{-2}} \end{pmatrix}. \end{aligned} \tag{53}$$

Having defined the generalised Wronskian as in equation (27), one gets

$$\begin{aligned} \lambda^2 \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\eta\begin{pmatrix}b\\c\end{pmatrix} V &= \lambda \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\begin{pmatrix} b-c(1+q') & 0 \\ 0 & -[b-c(1+q)] \end{pmatrix} V \\ &\quad + \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\eta\begin{pmatrix} -b_x - b(q'-r) \\ c_x - c(q-r') \end{pmatrix} V \end{aligned}$$

$$\begin{aligned} \lambda \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\eta & \begin{pmatrix} a(1+q)-d(1+q') \\ a-d \end{pmatrix} V \\ &= \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\begin{pmatrix} -a_x - a(q'-q) & 0 \\ 0 & -d_x - d(r'-r) \end{pmatrix} V \end{aligned}$$

$$\begin{aligned} \frac{1}{1+R'\bar{R}'} & \begin{pmatrix} 1+\bar{R}'R & -(\bar{R}'-\bar{R}) \\ R'-R & 1+\bar{R}R' \end{pmatrix} - \begin{pmatrix} TT'^{-1} & 0 \\ 0 & \bar{T}\bar{T}'^{-1} \end{pmatrix} \\ &= \lambda \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\eta\begin{pmatrix} -(q'-q) \\ 0 \end{pmatrix} V + \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\begin{pmatrix} q'-q & 0 \\ 0 & r'-r \end{pmatrix} V \end{aligned}$$

$$\begin{aligned} \frac{1}{1+R'\bar{R}'} & \begin{pmatrix} z^{-1}+z\bar{R}'R & z^{-1}\bar{R}-z\bar{R}' \\ z^{-1}R'-zR & z+z^{-1}\bar{R}R' \end{pmatrix} - \begin{pmatrix} z^{-1}TT'^{-1} & 0 \\ 0 & z\bar{T}\bar{T}'^{-1} \end{pmatrix} \\ &= \int_{-\infty}^{+\infty} \mathrm{d}x\, V'^{-1}\eta\begin{pmatrix} -q_x-(1+q)(q'-r) \\ r'-q \end{pmatrix} V. \end{aligned}$$

To construct the hierarchy of NEEs one uses equation (38) to get

$$\eta\lambda^2\begin{pmatrix}b\\c\end{pmatrix} = \eta \boldsymbol{L}\begin{pmatrix}b\\c\end{pmatrix}$$

$$\boldsymbol{L}\begin{pmatrix}b\\c\end{pmatrix} = \begin{pmatrix} -b_x + b(q+r) + 2b - 2c(1+q)^2 - 2[q_x+(1+q)(q-r)]\int_x^{\infty}\mathrm{d}\xi[b-c(1+q)] \\ c_x + c(q+r) + 2c - 2b - 2(q-r)\int_x^{\infty}\mathrm{d}\xi[b-c(1+q)] \end{pmatrix}$$

$$\frac{1}{\lambda}\begin{pmatrix} \dot{T}T^{-1}-\dfrac{\dot{R}\bar{R}}{1+R\bar{R}} & -\dfrac{\dot{\bar{R}}}{1+R\bar{R}} \\ \dfrac{\dot{R}}{1+R\bar{R}} & \dot{\bar{T}}\bar{T}^{-1}-\dfrac{\dot{\bar{R}}R}{1+R\bar{R}} \end{pmatrix} = \int_{-\infty}^{+\infty} \mathrm{d}x V^{-1}\eta\begin{pmatrix} -\dot{q}+(1+q)\int_x^{\infty}\mathrm{d}\xi(\dot{q}-\dot{r}) \\ \int_x^{\infty}\mathrm{d}\xi(\dot{q}-\dot{r}) \end{pmatrix} V$$

$$\frac{\mu}{1+R\bar{R}}\begin{pmatrix} R\bar{R} & -\bar{R} \\ -R & -R\bar{R} \end{pmatrix} = \int_{-\infty}^{+\infty} \mathrm{d}x V^{-1}\eta\begin{pmatrix} -q_x+(1+q)(r-q) \\ r-q \end{pmatrix} V.$$

Thus, if the reflection and transmission coefficients evolve in time according to the following linear equations

$$\dot{R}=\lambda\mu f(\lambda^2)R \qquad \dot{T}=0$$
$$\dot{\bar{R}}=-\lambda\mu f(\lambda^2)\bar{R} \qquad \dot{\bar{T}}=0$$

the 'potentials' q and r satisfy the following NEE

$$\begin{pmatrix} -\dot{q}+(1+q)\int_x^{\infty}\mathrm{d}\xi(\dot{q}-\dot{r}) \\ \int_x^{\infty}\mathrm{d}\xi(\dot{q}-\dot{r}) \end{pmatrix} + f(\boldsymbol{L})\begin{pmatrix} -q_x+(1+q)(r-q) \\ r-q \end{pmatrix} = 0. \tag{54}$$

The first system one obtains from equation (54) is for $f(\lambda^2)=1$

$$\dot{q}=-q_x \qquad \dot{r}=-r_x,$$

for $f(\lambda^2)=\lambda^2$

$$\dot{q}=q_{xx}-2(q_x+r_x)-2[q(r+q)]_x$$
$$\dot{r}=-r_{xx}-2(q_x+r_x)-2[r(r+q)]_x$$

and for $f(\lambda^2)=\lambda^4-4\lambda^2$

$$\begin{aligned} \dot{q}&=-q_{xxx}-(q^3+3qr^2)_x+3[(q_x-q-r)(q+r)]_x \\ \dot{r}&=-r_{xxx}-(r^3+3rq^2)_x-3[(r_x+q+r)(q+r)]_x. \end{aligned} \tag{55}$$

Equation (55) has, as a subcase for $r(x,t)=-q(x,t)$, the modified Korteweg–de Vries equation. As for the BTs, one has for $R'=z^2R$, $\bar{R}'=z^{-2}\bar{R}$, $T'=T$ and $\bar{T}'=\bar{T}$ the Volterra equation (48) and for $R'=z^4R$, $\bar{R}'=z^{-4}\bar{R}$, $T'=T$ and $\bar{T}'=\bar{T}$

$$q_x=(q'-q)\int_x^{\infty}\mathrm{d}\xi(q'-q+r'-r+q'r'-qr)-(r'-r)-(q'r'-qr)-q(q'-q)$$

$$r'_x=(r'-r)\int_x^{\infty}\mathrm{d}\xi(q'-q+r'-r+q'r'-qr)-(q'-q)-(q'r'-qr)-r'(r'-r)$$

i.e. a nonlocal BT.

4. The NDDEs of the discrete Zakharov–Shabat SP as BTs

Equation (9) is the compatibility condition between the SP (8) and the following equation which gives the evolution of the vector wavefunction $\boldsymbol{v}(n,t,z)$ of components

$(v_1(n,t,z), v_2(n,t,z))$:

$$\dot{\boldsymbol{v}}(n,t,z)=\begin{pmatrix} z-T(n-1,t)Q(n,t)+\Sigma_{k=-\infty}^{n-1}O(k,t) & Q(n,t)-z^{-1}S(n-1,t) \\ zT(n-1,t)+R(n,t) & z^{-1}-R(n,t)S(n-1,t)+\Sigma_{k=-\infty}^{n-1}O(k,t)\end{pmatrix}\boldsymbol{v}(n,t,z) \tag{56}$$

where $O(k,t)=T(k,t)[Q(k+1,t)-Q(k,t)]+S(k,t)[R(k+1,t)-R(k,t)]$.

In this case, the interpretation of equation (56) as a SP for which equation (9) is a BT, gives rise to a SP with an infinity of 'potentials', due to the infinite sums present in the diagonal terms of equation (56). Thus one leaves equation (9) as a non interesting case, going on to equation (10). Equation (10) is obtained as the compatibility condition between the following linear equations for the vector eigenfunction $\boldsymbol{v}(n,t,z)$

$$\boldsymbol{v}(n+1,t,z)=\begin{pmatrix} z & Q(n,t) \\ R(n,t) & z^{-1}\end{pmatrix}\boldsymbol{v}(n,t,z)$$

$$\dot{\boldsymbol{v}}(n,t,z)=\begin{pmatrix} z^2-Q(n,t)R(n-1,t) & zQ(n,t)+z^{-1}Q(n-1,t) \\ zR(n-1,t)+z^{-1}R(n,t) & z^{-2}-R(n,t)Q(n-1,t)\end{pmatrix}\boldsymbol{v}(n,t,z).$$

Changing the time variable into the x variable and introducing the four potentials $q(x)$, $r(x)$, $p(x)$ and $s(x)$ in the following way

$$\begin{aligned} q(x)&=Q(n,x) & p(x)&=Q(n-1,x)\\ r(x)&=R(n,x) & s(x)&=R(n-1,x)\\ q'(x)&=Q(n+1,x) & p'(x)&=Q(n,x)\\ r'(x)&=R(n+1,x) & s'(x)&=R(n,x)\end{aligned}$$

equation (10) reads

$$\begin{aligned} &q=p' & &r=s' \\ &q_x=(1-qr)(q'-p) & &r_x=(1-qr)(r'-s)\end{aligned} \tag{57}$$

and thus is a BT for the SP

$$\begin{aligned}\boldsymbol{v}_x(x,z)&=\begin{pmatrix} z^2-q(x)s(x) & zq(x)+z^{-1}p(x) \\ zs(x)+z^{-1}r(x) & z^{-2}-r(x)p(x)\end{pmatrix}\boldsymbol{v}(x,z)\\ &=U(q(x),p(x),r(x),s(x),z)\boldsymbol{v}(x,z).\end{aligned} \tag{58}$$

Introducing, as before, the variable $y=\mathrm{i}x$, the SP (58) reads

$$\boldsymbol{v}_y(y,z)=-\mathrm{i}U(q(y),p(y),r(y),s(y),z)\boldsymbol{v}(y,z). \tag{59}$$

As one chooses $q(y)$, $r(y)$, $p(y)$ and $s(y)$ to vanish at infinity, the Jost solutions for the SP (59) are given by equation (52). The Wronskian is defined by

$$\mathrm{Wr}(\boldsymbol{v}_1,\boldsymbol{v}_2)=\mathrm{i}\exp\left\{\mathrm{i}\left[(z^2+z^{-2})y+\int_y^\infty \mathrm{d}\xi(qs+rp)\right]\right\}\boldsymbol{v}_1^{\mathrm{T}}(y,z)\sigma_2\boldsymbol{v}_2(y,z)$$

and is constant in y. The asymptotics of the fundamental matrix solution of equation (59) is given by equation (53). Having defined the generalised Wronskian as in equation

(27), one gets

$$\lambda\mu\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\eta\begin{pmatrix}b\\c\end{pmatrix}V$$
$$=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\begin{pmatrix}b_x-b(rp-q's') & z(bs+cq')+z^{-1}(br+cp')\\ -z(bs'+cq)-z^{-1}(br'+cp) & -c_x-c(r'p'-qs)\end{pmatrix}V \tag{60}$$

$$\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\eta\begin{pmatrix}z(aq-dq')+z^{-1}(ap-dp')\\ -z(as'-ds)-z^{-1}(ar'-dr)\end{pmatrix}V$$
$$=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\begin{pmatrix}-a_x-a(q's'-qs) & 0\\ 0 & -d_x-d(r'p'-rp)\end{pmatrix}V \tag{61}$$

$$\frac{1}{1+\bar{R}'R'}\begin{pmatrix}1+\bar{R}'R & -[\bar{R}'-\bar{R}]\\ R'-R & R'\bar{R}+1\end{pmatrix}-\begin{pmatrix}TT'^{-1} & 0\\ 0 & \bar{T}\bar{T}'^{-1}\end{pmatrix}$$
$$=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\begin{pmatrix}q's'-qs & -z(q'-q)-z^{-1}(p'-p)\\ -z(s'-s)-z^{-1}(r'-r) & r'p'-rp\end{pmatrix}V \tag{62}$$

$$\frac{1}{1+\bar{R}'R'}\begin{pmatrix}1-\bar{R}'R & \bar{R}'+\bar{R}\\ R'+R & R'\bar{R}-1\end{pmatrix}-\begin{pmatrix}TT'^{-1} & 0\\ 0 & -\bar{T}\bar{T}'^{-1}\end{pmatrix}$$
$$=\int_{-\infty}^{+\infty}\mathrm{d}x\,V'^{-1}\begin{pmatrix}q's'-qs & z(q+q')+z^{-1}(p'+p)\\ -z(s'+s)-z^{-1}(r'+r) & -(r'p'-rp)\end{pmatrix}V. \tag{63}$$

Having set

$$\begin{aligned}
q(x)&=q(x,t) & q'(x)&=q(x,t)+\dot{q}(x,t)\,\mathrm{d}t\\
p(x)&=p(x,t) & p'(x)&=p(x,t)+\dot{p}(x,t)\,\mathrm{d}t\\
r(x)&=r(x,t) & r'(x)&=r(x,t)+\dot{r}(x,t)\,\mathrm{d}t\\
s(x)&=s(x,t) & s'(x)&=s(x,t)+\dot{s}(x,t)\,\mathrm{d}t\\
R(z)&=R(z,t) & R'(z)&=R(z,t)+\dot{R}(z,t)\,\mathrm{d}t\\
\bar{R}(z)&=\bar{R}(z,t) & \bar{R}'(z)&=\bar{R}(z,t)+\dot{\bar{R}}(z,t)\,\mathrm{d}t\\
T(z)&=T(z,t) & T'(z)&=T(z,t)+\dot{T}(z,t)\,\mathrm{d}t\\
\bar{T}(z)&\doteq\bar{T}(z,t) & \bar{T}'(z)&=\bar{T}(z,t)+\dot{\bar{T}}(z,t)\,\mathrm{d}t
\end{aligned}$$

one has

$$\eta\lambda\mu\begin{pmatrix}Az+Bz^{-1}\\ Cz+Dz^{-1}\end{pmatrix}=\eta\boldsymbol{L}\begin{pmatrix}Az+Bz^{-1}\\ Cz+Dz^{-1}\end{pmatrix}$$

$$\boldsymbol{L}\begin{pmatrix}Az+Bz^{-1}\\ Cz+Dz^{-1}\end{pmatrix}=\begin{pmatrix}[A_x+(A-2qY_2)(rp-qs)-2p(Y_1+Y_2)-2q(X-2X')-2(qY_2)_x]z\\ [-C_x-(C+2sY_2)(rp-qs)+2r(Y_1+Y_2)+2s(X-2X')-2(sY_2)_x]z\end{pmatrix.}$$
$$\begin{matrix}+[B_x+(B+2pY_1)(rp-qs)-2q(Y_1+Y_2)-2p(X-2X')+2(pY_1)_x]z^{-1}\\ +[-D_x-(D-2rY_1)(rp-qs)+2s(Y_1+Y_2)+2r(X-2X')+2(rY_1)_x]z^{-1}\end{matrix}$$

with

$$X = -\int_x^\infty d\xi (Ar + Bs + Cp + Dq) \qquad X' = -\int_x^\infty d\xi[(qr - ps)(Y_1 + Y_2)]$$

$$Y_1 = -\int_x^\infty d\xi (Br + Dp) \qquad Y_2 = -\int_x^\infty d\xi (As + Cq)$$

$$\begin{pmatrix} \dot{T}T^{-1} - \dfrac{\dot{R}\bar{R}}{1+R\bar{R}} & \dfrac{-\dot{\bar{R}}}{1+R\bar{R}} \\ \dfrac{\dot{R}}{1+R\bar{R}} & \dot{\bar{T}}\bar{T}^{-1} - \dfrac{\dot{\bar{R}}R}{1+R\bar{R}} \end{pmatrix} = \int_{-\infty}^{+\infty} dx V^{-1}\eta \begin{pmatrix} -z(\dot{q} - qX_0) - z^{-1}(\dot{p} - pX_0) \\ -z(\dot{s} + sX_0) - z^{-1}(\dot{r} + rX_0) \end{pmatrix} V$$

with

$$X_0 = -\int_x^\infty d\xi (\dot{r}p + \dot{p}r - \dot{q}s - q\dot{s})$$

$$\frac{1}{1+R\bar{R}}\begin{pmatrix} -R\bar{R} & \bar{R} \\ R & R\bar{R} \end{pmatrix} = \int_{-\infty}^{+\infty} dx V^{-1}\eta \begin{pmatrix} zq + z^{-1}p \\ -zs - z^{-1}r \end{pmatrix} V.$$

Thus, if the reflection and transmission coefficients evolve according to the following linear equation

$$\dot{R} = -f(\lambda\mu)R \qquad \dot{T} = 0 \qquad \dot{\bar{R}} = f(\lambda\mu)\bar{R} \qquad \dot{\bar{T}} = 0$$

then the 'potentials' evolve according to the following NEE

$$\begin{pmatrix} z(\dot{q} - qX_0) + z^{-1}(\dot{p} - pX_0) \\ z(\dot{s} + sX_0) + z^{-1}(\dot{r} + rX_0) \end{pmatrix} = f(\boldsymbol{L})\begin{pmatrix} zq + z^{-1}p \\ -zs - z^{-1}r \end{pmatrix}. \tag{64}$$

The simplest elements of equation (64) are for $f(\lambda\mu) = 1$

$$\dot{q} = q \qquad \dot{p} = p \qquad \dot{r} = -r \qquad \dot{s} = -s,$$

for $f(\lambda\mu) = \lambda\mu$†

$$\begin{aligned} \dot{q} &= q_x + 2q(rp - qs) - 2q\int_x^\infty d\xi[rp - qs]^2 \\ \dot{p} &= p_x + 2p(rp - qs) - 2p\int_x^\infty d\xi[rp - qs]^2 \\ \dot{r} &= r_x + 2r\int_x^\infty d\xi[rp - qs]^2 \\ \dot{s} &= s_x + 2s\int_x^\infty d\xi[rp - qs]^2. \end{aligned} \tag{65}$$

As for the BTs, from equations (62) and (63), applying equations (60) and (61) one arrives, after some nontrivial calculations, at equation (57) corresponding with the following relation between the reflection and transmission coefficients:

$$\bar{R}' = z^2\bar{R} \qquad \bar{T}' = \bar{T} \qquad R' = z^{-2}R \qquad T' = T.$$

† It is worthwhile to notice that, for $u = (rp - qs)$, from equation (65) one has: $\dot{u} = u_x + 2u^2$.

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